The Simple AR Model as an Infinite-Lag MA Model

We already examined how an Infinite-lag AR model with some restrictions is equivalent to a simple MA model, but what about the opposite?

In this short lecture, we are going to showcase another relationship between autoregressive and moving-average models. We will try to show how a simple AR model is equivalent to an infinite-lag MA model with certain restrictions.

Before we begin, we want to state that to fully follow this solution, you would need to know some Calculus. We will try to be as clear as possible for those who do not have the mathematical background, but please look up any unfamiliar notation while going through the slides.

Let's begin!

$$AR(1) = MA(\infty)$$

Simple AR

Infinite-Lag MA Model
(with certain restrictions)

$MA(\infty)$ Model

We'll start by elaborating what we mean by "infinite"-lag MA model: $MA(\infty)$

By "infinite"-lag MA model, we mean a moving-average model, which takes the error terms of *infinitely* many previous periods. Of course, just like the "infinite"-lag AR model, this is just a **theoretical** concept.

We cannot have an **infinite** data set, even when working with big data. Therefore, no data set would ever satisfy the information criteria required for this kind of model. So, an infinite lag autoregressive model is not applicable in practice.

However, we can write out what modelling such a variable would look like. Suppose "r" is a variable, which follows this model. Then, the $MA(\infty)$ looks like this:

Parameter Coefficients

Residuals

$$r_t = c + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \epsilon_t$$

Value of Interest

Constant

Past Values

$AR(1)$ Model

Now, that you've already seen the infinite-lag moving-average model, let's go back to the simple AR model.

Once again, suppose our variable of interest is "r", so the model would looks something like this:

$$r_t = c + \phi r_{t-1} + \epsilon_t$$

To avoid confusion between the two models, let's call the constant for the AR model, c_{AR} (no, not the vehicle) and the one for the MA model - c_{MA}.

Then, the proper expression of the AR(1) would be as follows:

$$r_t = c_{AR} + \phi r_{t-1} + \epsilon_t$$

Now that the ambiguity is settled, let's talk about expressing past values.

Past and Present Values

By the definition of AR Models, we can express r_{t-1} in a similar way to r_t. More precisely, if the model suggests that $r_t = c_{AR} + \phi r_{t-1} + \epsilon_t$, then, clearly, we can conclude the following.

$$r_{t-1} = c_{AR} + \phi r_{t-2} + \epsilon_{t-1}$$

Then, we can plug this value into the r_t equation to get:

$$r_t = c_{AR} + \phi(c_{AR} + \phi r_{t-2} + \epsilon_{t-1}) + \epsilon_t$$

Of course, we can expand the expression in parentheses and re-arrange to get:

$$r_t = c_{AR} + \phi c_{AR} + \phi^2 r_{t-2} + \phi \epsilon_{t-1} + \epsilon_t$$

Déjà Vu: Past and Present Values

We can express r_{t-2} in a similar way to r_{t-1} and substitute it into our new equation.

$$r_{t-2} = c_{AR} + \phi r_{t-3} + \epsilon_{t-2}$$

Then, we can plug this value into the r_t equation to get:

$$r_t = c_{AR} + \phi c_{AR} + \phi^2 (c_{AR} + \phi r_{t-3} + \epsilon_{t-2}) + \phi \epsilon_{t-1} + \epsilon_t$$

Of course, we can (once again) expand the expression in parentheses and re-arrange to get:

$$r_t = c_{AR} + \phi c_{AR} + \phi^2 c_{AR} + \phi^3 r_{t-3} + \phi^2 \epsilon_{t-2} + \phi \epsilon_{t-1} + \epsilon_t$$

Generalizing the Equations

We can observe the last to version of the r_t equations and find a pattern that can be generalized.

$$r_t = (c_{AR} + \phi c_{AR}) + \phi^2 r_{t-2} + (\phi \epsilon_{t-1} + \epsilon_t)$$

$$r_t = (c_{AR} + \phi c_{AR} + \phi^2 c_{AR}) + \phi^3 r_{t-3} + (\phi^2 \epsilon_{t-2} + \phi \epsilon_{t-1} + \epsilon_t)$$

We can use the summation expression to generalize this for more than 3 periods back and get:

$$r_t = (c_{AR} + \phi c_{AR} + \cdots + \phi^k c_{AR}) + \phi^k r_{t-k} + (\phi^k \epsilon_{t-k} + \cdots + \phi \epsilon_{t-1} + \epsilon_t)$$

Generalizing the Equations pt. 2

Since c_{AR} is a constant, we can take it outside the parentheses and transform the equation as such:

$$r_t = c_{AR}(1 + \phi + \cdots + \phi^k) + \phi^k r_{t-k} + (\phi^k \epsilon_{t-k} + \cdots + \phi\epsilon_{t-1} + \epsilon_t)$$

We know the equation holds true for k-many lags, so what if we had an infinitely many lags. Then r_t would look like:

$$r_t = c_{AR}(1 + \phi + \cdots + \phi^\infty) + \phi^\infty r_{t-\infty} + (\phi^\infty \epsilon_{t-\infty} + \cdots + \phi\epsilon_{t-1} + \epsilon_t)$$

Before we proceed, let's elaborate on the notation for a second. The $r_{t-\infty}$ and $\epsilon_{t-\infty}$ notations suggest values and errors infinitely-many periods ago. Of course, this is only theoretical and we use it to express the **initial** (first period) values.

Infinite Sums

We can now use sigma to represent the summation of the coefficients multiplied by the constant, so we get:

$$r_t = c_{AR} \sum_{m=0}^{\infty} \phi^m + \phi^{\infty} r_{t-\infty} + (\phi^{\infty} \epsilon_{t-\infty} + \cdots + \phi \epsilon_{t-1} + \epsilon_t)$$

Since $|\phi| < 1$, we know how to solve the infinite sum $\sum_{m=0}^{\infty} \phi^m$ of a geometric series. It simply equals $\frac{1}{1-\phi}$. Then, we can substitute this into the equation to get:

$$r_t = c_{AR} \frac{1}{1-\phi} + \phi^{\infty} r_{t-\infty} + (\phi^{\infty} \epsilon_{t-\infty} + \cdots + \phi \epsilon_{t-1} + \epsilon_t)$$

Model Restrictions

Next up, since MA models don't rely on past values, but only past residuals, we want to get rid off the $r_{t-\infty}$. Luckily for us $|\phi| < 1$, so ϕ^{∞}converges to 0.

$$r_t = c_{AR}\frac{1}{1-\phi} + (\phi^{\infty}\epsilon_{t-\infty} + \cdots + \phi\epsilon_{t-1} + \epsilon_t)$$

Okay, we are almost done. If you look closely, we have an equation that consists of a constant, a residual and past residuals multiplied by certain coefficients. These are all the pasts included in an MA model. Thus, if we put some restrictions like $\theta_k = \phi^k$we get:

$$r_t = c_{AR}\frac{1}{1-\phi} + \theta_{\infty}\epsilon_{t-\infty} + \cdots + \theta_1\epsilon_{t-1} + \epsilon_t$$

Substitution and Rearranging the Model

Since $c_{AR}\frac{1}{1-\phi}$ is just a number, we can substitute it with another constant like c_{MA}.

$$r_t = c_{MA} + \theta_\infty \epsilon_{t-\infty} + \cdots + \theta_1 \epsilon_{t-1} + \epsilon_t$$

Additionally, we can rearrange the past errors and their associated coefficients and get the following equation:

$$r_t = c_{MA} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_\infty \epsilon_{t-\infty} + \epsilon_t$$

If we have a quick look, we can clearly see that this is the exact same representation we wrote down for an infinite-lag MA model earlier. The only difference comes from us specifically defining the constant, since we had two – one for the AR and another one for the MA.

$MA(\infty)$ Model

And here we can examine this new representation of the model.

Constant

Coefficients

$$r_t = c_{MA} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \epsilon_t$$

Value of Interest

Past Errors

Residual

We have seen how an infinite AR model with some restrictions is equivalent to an MA(1) model, and we just saw how an simple AR model is also equivalent to an infinite MA model. These two relationships between the models are very unique and make it all the more difficult to determine the correct order, when we want to use both simultaneously (the ARMA model).